

A Dark Energy Model

Characterized by the Age of the Universe

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Abstract

Quantum mechanics together with general relativity leads to the Károlyházy relation and a corresponding energy density of quantum fluctuations of space-time. Based on the energy density we propose a dark energy model, in which the age of the universe is introduced as the length measure. This dark energy is consistent with astronomical data if the unique numerical parameter in the dark energy model is taken to be a number of order one. The dark energy behaves like a cosmological constant at early time and drives the universe to an eternally accelerated expansion with power-law form at late time. In addition, we point out a subtlety in this kind of dark energy model.

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Needless to say, the cosmological constant problem is one of the biggest challenges in theoretical physics [1]. Naive estimation leads the cosmological constant to be of the Planck scale $(10^{19}GeV)^4$; if SUSY breaks at TeV scale, the cosmological constant should be in the order $(1TeV)^4$. The discovery of the current accelerated expansion of the universe causes the problem to be more difficult to solve [2], which implies that the cosmological constant is in the scale $(10^{-3}ev)^4$. There exists a big hierarchy difference between the theoretical estimation and observation value. Since the cosmological constant is related to the vacuum expectation value of some quantum fields and it can be measured only through gravitational experiments. Therefore the cosmological constant problem is essentially a problem in quantum gravity. Although a completely successful quantum theory of gravity is still not yet available, quantum mechanics together with general relativity may shed some lights on this issue.

General relativity tells us that any classical physical laws concerning space-time can be verified without any limit in accuracy. To make a measurement of space-time, one has to introduce an experiment device. However, there exists a well-known Heisenberg uncertainty relation in quantum mechanics. The Heisenberg uncertain relation combining with general relativity leads to a fundamental scale of microstructure of space-time: Planck length $l_p \sim 10^{-33}cm$. Following the line of quantum fluctuations of space-time, Károlyházy and his collaborators [3] made an interesting observation concerning the distance measurement for Minkowski space-time through a light-clock *Gedankenexperiment* (see also [4]): The distance t in Minkowski space-time cannot be known to a better accuracy than

$$\delta t = \beta t_p^{2/3} t^{1/3}, \quad (1)$$

where β is a numerical factor of order one, t_p is the reduced Planck time, and throughout this paper, we use the units $c = \hbar = k_b = 1$, so that one has $l_p = t_p = 1/m_p$ with l_p and m_p being the reduced Planck length and mass, respectively.

The Károlyházy relation (1) together with the time-energy uncertainty relation enables one to estimate a quantum energy density of the metric fluctuations of Minkowski space-time [4, 5]. With the relation (1), a length scale t can be known with a maximal precision δt determining a minimal detectable cell δ^3 over a spatial region t^3 . Thus one is able to look at the microstructure of space-time over a region t^3 by viewing the region as the one consisting of cells $\delta t^3 \sim t_p^2 t$. Therefore such a cell δt^3 is the minimal detectable unit of space-time over a given length scale t and if the age of the space-time is t , its existence due to the time-energy uncertainty relation cannot be justified with energy smaller than $\sim t^{-1}$. Hence, as a result of the relation (1), one can conclude that if the age of the Minkowski space-time is t over a spatial region with linear size t (determining the maximal observable

patch) there exists a minimal cell δt^3 , the energy of the cell cannot be smaller than [5]

$$E_{\delta t^3} \sim t^{-1}, \quad (2)$$

due to the time-energy uncertainty relation. And then the energy density of the metric fluctuations of the Minkowski space-time is [5]

$$\rho_q \sim \frac{E_{\delta t^3}}{\delta t^3} \sim \frac{1}{t_p^2 t^2}. \quad (3)$$

The existence of this energy is necessary to ensure the stability of the background space-time against the metric fluctuations since the relation (1) determines the maximal accuracy allowed by the nature [5]. More recently Maziashvili [5] has investigated the cosmological implications of the Károlyházy relation (1) and the energy density (3) in the different stages of cosmological evolutions including inflation epoch, radiation, matter and dark energy dominated phases, respectively. It was found that to be consistent, $\beta^3 \approx 32\pi/3$, $72\pi/12$, and $8\pi/3$ during the radiation, matter and dark energy dominated phases, respectively. The Károlyházy relation (1) and the energy density (3) have also been discovered independently in [6, 7]. At this stage, we would like to mention that there exist some controversies on the validness of the Károlyházy relation (1) in literature, see, for example, [8, 9] and references therein.

Here some remarks are in order on the Károlyházy relation (1) and the energy density (3). First, let us mention that the Károlyházy relation (1) obeys the holographic black hole entropy bound [5]: the relation (1) gives a relation between an UV cutoff δl and the length scale l of a system, $\delta l \sim l_p^{2/3} l^{1/3}$; the system has entropy

$$S \leq \left(\frac{l}{\delta l} \right)^3 \sim \left(\frac{l}{l_p} \right)^2 \sim S_{\text{BH}}, \quad (4)$$

which is less than the black hole entropy with horizon radius l . Therefore, the Károlyházy relation (1) is a reflection of entanglement between UV scale and IR scale in effective quantum field theory [10].

Second, the authors of [10] argued that considered the effect of gravity, the vacuum energy density ρ_Λ of a certain effective quantum field in a finite region with length scale l cannot be arbitrary large, otherwise the region will collapse to a black hole with size l . This implies that $\rho_\Lambda l^3 \leq l/l_p^2$, which leads to

$$\rho_\Lambda \sim \frac{1}{l_p^2 l^2}. \quad (5)$$

One immediately sees that the energy density (3) has the same form as the one (5), the so-called holographic energy density, although the energy density (3) describes quantum

fluctuations of Minkowski space-time. The similarity between (3) and (5) might reveal some universal feature of quantum gravity since one arrives at (3) and (5) both by considering quantum effect of gravity, albeit in different way.

Third, let us mention that the cosmological implications of the holographic energy (5) has been investigated intensively. Choosing the Hubble horizon $1/H$ of the universe as the length scale l in (5), the holographic energy (5) indeed gives the observation value of dark energy in the universe. However, as found by Hsu [11], in that case, the evolution of the dark energy is the same as that of dark matter (dust matter), and therefore it cannot drive the universe to accelerated expansion. The same appears if one chooses the particle horizon of the universe as the length scale l [12]. An interesting proposal is made by Li [12]: Choosing the event horizon of the universe as the length scale, the holographic dark energy (5) not only gives the observation value of dark energy in the universe, but also can drive the universe to an accelerated expansion phase. In that case, however, an obvious drawback concerning causality appears in this proposal. Event horizon is a global concept of space-time; existence of event horizon of the universe depends on future evolution of the universe; and event horizon exists only for universe with forever accelerated expansion. In addition, more recently, it has been argued that this proposal might be in contradiction to the age of some old high redshift objects, unless a lower Hubble parameter is considered [13] (by the way, a complete list of references concerning the holographic dark energy can be found in [13]).

In this note we propose a dark energy model based on the energy density (3). The big difference from (5) is that we choose the age of the space-time as the length measure, instead of the horizon distance of the universe. Thus the causality problem in the holographic dark energy is avoided. Note that energy density for quantum fluctuations of matters in the universe has the same order as the one (3) for the metric fluctuation. We introduce a numerical factor n^2 to parameterize some uncertainties, for example, the species of quantum fields in the universe, the effect of curved space-time (since the energy density is derived for Minkowski space-time), etc. As a result, we write down the energy density of quantum fluctuations in the universe as

$$\rho_q = \frac{3n^2 m_p^2}{T^2}, \quad (6)$$

as the dark energy in our universe, where T is the age of the universe, and the introduction of the number 3 is for later convenience.

The energy density (6) with the current age of the universe, $T \sim 1/H_0$ (here H_0 is the current Hubble parameter of the universe), explicitly gives us the observed dark energy density, provided the numerical factor n is of order one (Turn the logic around,

the parameter n can be estimated by observational data in Λ CDM model [14]: $H_0 = 72 \text{ km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}$, $T = 13.7 \text{ Gyr}$ and $\Omega_{de} = 0.73$, then one has $n = 1.15$). Next, let us see whether the energy density (6) can drive the universe to accelerated expansion. For the sake of simplicity, we first consider the case without other matter in the universe. In this case, the Friedmann equation for a flat FRW universe is

$$H^2 = \frac{1}{3m_p^2} \rho_q, \quad (7)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter, a is the scale factor of the universe and the overdot stands for derivative with respect to the cosmic time. The age of the universe can be calculated through

$$T = \int_0^a \frac{da}{Ha}. \quad (8)$$

Solving the Friedmann equation (7) yields the evolution of the universe with the scale factor

$$a = [n(H_0 t + \alpha)]^n, \quad (9)$$

where α is an integration constant, which can be determined by assuming the present scale factor $a_0 = 1$. We can see clearly from (9) that the universe is in the accelerated expansion phase provided $n > 1$. The equation of state for the energy density (6) turns out to be

$$w_q = -1 + \frac{2}{3n}. \quad (10)$$

Indeed one can see from (10) that the energy density can drive the universe to accelerated expansion if $n > 1$. In addition, let us stress an interesting point here that without any inflaton, the energy density (6) of quantum fluctuations can give rise to an inflationary period in the early universe.

Now let us consider the case with dark (dust) matter in the universe. In this case, the corresponding Friedmann equation is

$$H^2 = \frac{1}{3m_p^2} (\rho_m + \rho_q). \quad (11)$$

Defining the fraction energy density of dark matter as $\Omega_m = \rho_m/3m_p^2 H^2$, and $\Omega_q = \rho_q/3m_p^2 H^2$ for the dark energy, one has $\Omega_q = n^2/T^2 H^2$. Using the Friedmann equation (11), we get the equation of motion for Ω_q as

$$\Omega'_q = \frac{2}{n} \left(\frac{3n}{2} - \sqrt{\Omega_q} \right) (1 - \Omega_q) \Omega_q, \quad (12)$$

where the prime represents the derivative with respect to $\ln a$. This equation can be integrated analytically, one has

$$\begin{aligned} \frac{1}{n} \ln a + c_0 = & -\frac{1}{3n-2} \ln(1 - \sqrt{\Omega_q}) - \frac{1}{3n+2} \ln(1 + \sqrt{\Omega_q}) \\ & + \frac{1}{3n} \ln \Omega_q + \frac{8}{3n(9n^2-4)} \ln\left(\frac{3n}{2} - \sqrt{\Omega_q}\right). \end{aligned} \quad (13)$$

where c_0 is an integration constant, which can be determined by current observations, for example, WMAP with $\Omega_{q0} = 0.73$ as $a = 1$ [14]. Although the expression (13) is not instructive, it is easy to see that the fraction energy density Ω_q indeed decreases when it goes back to early time. To see the evolution behavior of the dark energy density, let us study its behavior in two different stages. The first one is the matter dominated phase, where $a \sim 0$ and $\Omega_q \sim 0$. In this case, we have the solution to the equation (12)

$$\Omega_q \approx c_1 a^3, \quad (14)$$

where c_1 is another integration constant. The fraction dark energy density increases during the epoch of matter domination. The other is the dark energy dominated phase, where $\Omega_m \sim 0$ and $\Omega_q \sim 1$. In that case, we get the solution to the equation (12)

$$\Omega_q \approx 1 - c_2 a^{-(3n-2)/n}, \quad (15)$$

where c_2 is an integration constant. We see from (14) and (15) that the fraction dark energy density increases quickly and is independent of the parameter n at earlier time of matter dominated phase, while it approaches to one in a manner depending on n in the dark energy dominated phase at later time.

The equation of state for the dark energy can be easily obtained through the formula, $w_q = -1 - \dot{\rho}_q / (3H\rho_q)$. It gives us with

$$w_q = -1 + \frac{2}{3n} \sqrt{\Omega_q}. \quad (16)$$

Once given the fraction energy density, the current equation of state is completely determined by the parameter n . In Fig. 1 we plot the current equation of state with respect to the parameter n , provided $\Omega_{q0} = 0.73$. We see that $w_{q0} \leq -0.81$ as $n \geq 3$. Therefore the equation of state is consistent with the WMAP observation [14], as the parameter n is taken to be a number of order one.

The equation of state (16) has an interesting feature. At earlier time where $\Omega_q \rightarrow 0$, one has $w_q \rightarrow -1$. Namely, the dark energy behaves like a cosmological constant at earlier time. At later time where $\Omega_q \rightarrow 1$, the equation of state (16) goes back to the case (10).

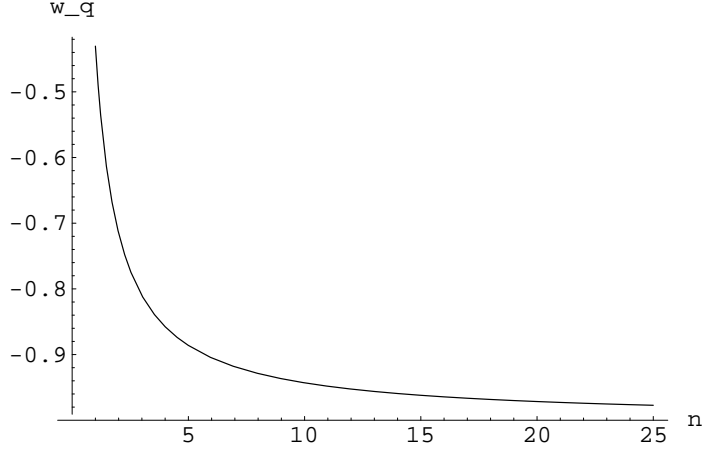


Figure 1: This plot shows the current equation of state for the dark energy versus the parameter n , provided $\Omega_{q0} = 0.73$.

Therefore the fate of our universe is an eternally accelerated expansion with power-law form (9) in this dark energy model.

After a close look at (14) and (16), a confusion arises. In the matter dominated phase, if the dark energy is negligible and $a \sim t^{2/3}$, one then has $\Omega_q = 9n^2/4$. This is obviously in contradiction to (14). A more close look at the equation (12) tells us that this equation not only holds for the form $T = \frac{n}{H\sqrt{\Omega_q}}$, but also for another form $T' = T + \delta = \frac{n}{H\sqrt{\Omega_q}}$, where δ is a constant, because the equation (12) is obtained by taking derivative of the form $T = \frac{n}{H\sqrt{\Omega_q}}$ with respect to the cosmic time, together with the energy conservation equation (or taking derivative with respect to the cosmic time on both sides of equation $\int_0^a \frac{da}{Ha} = \frac{n}{H\sqrt{\Omega_q}}$). As a result, the cosmic age T' obtained by solving (12) or using (13) might be different from the one (8) by a constant δ . The constant δ can be determined by $\delta = \frac{n}{H\sqrt{\Omega_q}} - \int_0^a \frac{da}{Ha}$. Clearly the constant δ depends on the parameter n as well as the current Hubble parameter H_0 and fraction energy density (Ω_{m0}) of dark matter (or equivalently, the fraction dark energy density Ω_{q0})¹. Thus we can easily understand the results (12) and (16). When $T \ll \delta$ at earlier time in the matter dominated phase, the dark energy behaves as a cosmological constant, while it drives the universe to an eternally

¹A similar situation occurs for the holographic dark energy [12]. In that model, there exist two expressions for the horizon distance: $R'_h = \frac{c}{H\sqrt{\Omega_{de}}}$ and $R_h = a \int_a^\infty \frac{da}{Ha^2}$. The former can be obtained by solving a similar equation as (12) by adding the initial condition for the current observational data. Nothing can guarantee these two expressions are equal and a constant difference between them exists. In fact, the same situation appears in the similar models.

accelerated expansion in a power-law form (9) at later time.

Another approach to way out the confusion is to consider n in (6) as a slowly varying function of the age of the universe. For example, the parameter n might be dependent of the cosmic age in some way: in the early stage, it changes the form (6) in some manner so that n is negligible small (for instance, $n \sim T$) and at some time, it approximately turns to be a constant. Indeed, the energy form (3) is derived from an argument in Minkowski spacetime. In the early universe, where the space is highly curved and dynamical, it is conceivable to think out that the parameter n depends on the age of the universe in some way.

To summarize, we have proposed a dark energy model based on the Károlyházy relation (1), and energy density of quantum fluctuations of matter and metric in the universe. The dark energy density (6) has the same form as the holographic dark energy, but we have introduced the age of the universe as the length measure, instead of the horizon distance of the universe. Thus the causality problem in the holographic dark energy is avoided. Our dark energy model not only gives the observed value of dark energy in the universe, but also can drive the universe to accelerated expansion. Its equation of state can be consistent with astronomical data, provided the unique parameter in the dark energy is taken to be a number of order one. In this model, the dark energy behaves like a cosmological constant at early time and it drives the universe to an eternally accelerated expansion with power-law form (9) at later time.

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